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Conjugation in Representation Categories of Multiplicative Unitaries and Their Actions on C^*-Algebras

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A conjugation functor $\mathscr{F}$ on a full subcategory of $\mathscr{R}(V)$, the representation category of a multiplicative unitary V, is defined. If V has a conjugate, it is also regular and the domain of $\mathscr{F}$ is all $\mathscr{R}(V)$. Examples of selfconjugate multiplicative unitaries are discussed. A coaction of the Hopf C^*-algebra associated with V on the Cuntz algebra $\mathscr{O}_d$ is canonically defined by a unitary object W of $\mathscr{R}(V)$ acting on a d-dimensional Hilbert space. As in the group action case if $d=\infty$ and W belongs to the domain of $\mathscr{F}$, ergodic coactions are often characterized by the absence of finite dimensional subrepresentations of W. Furthermore model actions of compact quantum group duals on C^*-algebras are defined. © 1996 Academic Press, Inc.

1. INTRODUCTION

If G is a compact group then any continuous unitary finite dimensional representation u of G on a Hilbert space H contained in an ambient von Neumann algebra M, induces an automorphic action α on $C^*(H)$, the C^*-algebra generated by H [1]; the corresponding fixed point algebra $C^*(H)_\alpha$ carries a natural action of a dual object of G. More specifically there is a faithful functor from the category of representations of G with objects the tensor powers of u to $\mathrm{End}(C^*(H)_\alpha)$, the tensor C^*-category of endomorphisms of the fixed point algebra. The image of that functor is a full subcategory $\mathscr{T}_\sigma$ with objects the powers of a single endomorphism σ defined as the restriction to the fixed points of the inner endomorphism of $C^*(H)$ induced by H [2]. The basic property of commutativity of the tensor product between representations is reflected in the property that σ has permutation symmetry.

If H has left support the identity operator I of M then $C^*(H)$ coincides with the Cuntz algebra $\mathscr{O}_d$ [1] of order the dimension of H. If in particular

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u is the defining representation of a compact Lie group $G \subseteq SU(H)$ then the corresponding dual action on the fixed point algebra $\mathcal{O}_G$, regarded in [3] as a model action, can be characterized intrinsically [4]. This basic result later led to a new duality theorem for compact groups characterizing representation categories of compact groups as abstract categories and to the solution of a long-standing problem in Quantum Field Theory [4], [5], [6].

An alternative point of view is that of considering the canonical action induced by the left regular representation λ of a compact group G. If G is finite with d elements the fixed point algebra is isomorphic to the whole Cuntz algebra. This result generalizes in fact to amenable locally compact groups acting on the generalized Cuntz algebra [7]. In general there is again a faithful functor from the whole finite dimensional representation category of G to a full permutation symmetric subcategory $\mathscr{T}$ of $\mathrm{End}(C^*(H)_\alpha)$ with conjugates. The objects of $\mathscr{T}$ are the restrictions to the fixed points of the inner endomorphisms of $C^*(H)$ induced by the finite dimensional G-submodules of the powers of H. Such actions have been intrinsically characterized in [8].

A question then naturally arises. Does the above duality theory generalize to locally compact groups or, more generally, to locally compact quantum groups? It has been shown in [9] that if G is allowed to be non-compact and H is infinite dimensional then the fixed point algebra $C^*(H)_\alpha$ depends only on the finite dimensional G-submodules of H, so it is too thin to accomodate a reasonable duality theory for locally compact groups. One has to consider, instead, canonical group actions induced by the Hilbert space representations on the generalized Cuntz algebra $\mathcal{O}_H$ in order to get model actions of G-duals on C^*-algebras. These topics, together with further remarks on duality theory for locally compact groups and possible generalizations to quantum groups can be found in [7].

In this paper we want to suggest a setting for an abstract duality theorem for compact quantum groups in the framework of C^*-algebras.

In their recent work S. Baaj and G. Skandalis describe Hopf C^*-algebras $(\mathscr{A}, \delta)$ in terms of suitable unitary operators V on Hilbert spaces, that they call multiplicative. In their formalism, representations of $(\mathscr{A}, \delta)$ correspond to corepresentations of V.

This suggests how to define actions of certain Hopf algebras on the C^*-algebras $C^*(H)$. More specifically if V is a regular multiplicative unitary on some Hilbert space $K^2 = K \otimes K$ then any unitary corepresentation W of V on H induces a coaction λ of the Hopf C^*-algebra $\hat{\mathscr{A}}(V)$ coming from the first component of V on $C^*(H)$ via the monomorphism defined by the unitary $W\vartheta_{H,K}$ from $C^*(H)$ to M. Here $\vartheta_{H,K}\colon H \otimes K \to K \otimes H$ denotes the operator that interchanges the order of H and K.

If V acts on a finite dimensional Hilbert space then the action corresponding to the regular corepresentation was first defined by J. Cuntz in [11].

If V is the standard multiplicative unitary associated to a locally compact group G, then any corepresentation W corresponds to a unitary representation of G and the induced coaction to the corresponding canonical automorphic action above discussed. The canonical regular action corresponds to the regular corepresentation V. The group G is compact if and only if V is of discrete type. The following results generalize results previously known in the group action case.

If λ corresponds to the regular corepresentation and K is finite dimensional then the fixed point algebra $C^*(K)_\lambda$ is again isomorphic to $C^*(K)$ (as in the group action case, this result has a generalization in the framework of generalized Cuntz algebras [7]). This fact has been proved independently by R. Longo in [12].

If V is regular and of discrete type, that is it comes from a Woronowicz C^*-algebra, and K is infinite dimensional then the corresponding fixed point algebra contains $C^*(K)$ in a canonical fashion, and it is a simple, nuclear and purely infinite.

We give a notion of conjugate object of any corepresentation of a multiplicative unitary that generalizes the previously known one for a Woronowicz C^*-algebra [17].

We show that if the regular corepresentation has a conjugate then the underlying multiplicative unitary V is regular and moreover any other W has a conjugate. Any such conjugate is unitary if and only if the natural densely defined coinverse map of $\hat{\mathscr{A}}(V)$ is involutive.

If V is of discrete type then the notion of regularity is in fact equivalent to that of admitting a conjugate object and, in this case, the regular corepresentation is selfconjugate. Now the conjugate of any finite dimensional corepresentation is equivalent to a unitary corepresentation, unique up to equivalence, that may be characterized intrinsically. This characterization coincides exactly with the notion of conjugate object in the framework of tensor C^*-categories [7].

As in the group action case any tensor category of unitary finite-dimensional corepresentations of V can be embedded in a full subcategory of $\mathrm{End}(C^*(K)_\lambda)$ via a faithful functor, where λ corresponds to the regular corepresentation. There is no permutation symmetry in general since the tensor product between corepresentations is not commutative. However, there are interesting examples gifted with a sufficiently weak notion of braided symmetry [10], [14], [7].

This sets the stage for a duality theorem for compact quantum groups. The next step should be that of characterizing those full tensor C^*-subcategories with conjugates of some $\mathrm{End}(\mathscr{A})$, $\mathscr{A}$ a C^*-algebra, that come from the finite dimensional representation category of a necessarily regular multiplicative unitary of discrete type.

If W is a corepresentation of a regular multiplicative unitary V then the fixed point subalgebra $C^*(H)_\lambda$ is generated, as a Banach space, by the subspaces of compact interwiners between the tensor powers of W. If in particular W has a conjugate equivalent to a unitary corepresentation then $C^*(H)_\lambda$ reduces to the complex numbers unless W has a non-trivial finite dimensional subcorepresentation and this generalizes the result previously known in the group action case.

2. Preliminaries

We need to recall from [10] some of the main definitions involving multiplicative unitaries and their representations. In order to simplify notation we choose to work in the strict tensor W^*-category [13] $\mathscr{H}(M)$ of Hilbert spaces in some ambient von Neumann algebra M. Recall that a tensor W^*-category $\mathscr{T}$ is called strict if the tensor product defined on the objects and on the arrows is strictly associative. This implies in particular that the set of objects of $\mathscr{T}$ is a unital semigroup.

An object of $\mathscr{H}(M)$ is a norm closed subspace H of M such that for any pair of vectors $\psi, \varphi \in H$, $\psi^*\varphi$ is a complex multiple of the identity operator I. This scalar then defines an inner product on H. If A and B are Banach subspaces of M we write AB for the norm closed subspace of M generated by the operator products of elements of A and B.

If H, H' are Hilbert spaces of M, $\mathscr{K}(H, H') := H'H^*$ is identified with the compact operators from H to H'. Similarly its ultra-weak closure (H, H') is identified with the set of all bounded linear operators from H to H' and is defined as the set of arrows from H to H' in $\mathscr{H}(M)$.

The strict tensor product is defined on the objects and on the arrows respectively by

$$H \times H' = HH',$$

with unit object $\mathbb{C}I$, and

$$R \times S = R\sigma_H(S) = \sigma_{H'}(S)\, R \in (HK, H'K'), \qquad R \in (H, H'),\ S \in (K, K') \tag{2.1}$$

where σ_H denotes the inner endomorphism of M induced by H. $\mathscr{H}(M)$ is clearly isomorphic to a full subcategory of $\mathscr{H}$, the category of Hilbert spaces, as a tensor W^*-category. Now every full W^*-subcategory of $\mathscr{H}$ such that the dimensions of its objects are all bounded by a fixed cardinal can be realized as a full subcategory of some $\mathscr{H}(M)$ [14].

This allows us to realize every Hilbert space we need as a Hilbert space in M and to use M as an ambient algebra to work in. If H is a Hilbert space in M with support $\sigma_H(I)$ equal to the identity operator I, we define $\mathcal{O}_H$ as the closed subspace generated by the subspaces (H^r, H^s), $r, s \in \mathbb{N}_0$. This is a simple C^*-algebra and coincides with the Cuntz algebra of order d if H has finite dimension d [1], [2], [7].

If $\sigma_H(I)$ is a proper projection, the analogous definition yields a non-simple C^*-algebra $\mathscr{E}_H$ with a unique ideal $\mathscr{I}$ isomorphic to the compact operators on H and the quotient C^*-algebra $\mathscr{E}_H/\mathscr{I}$ is isomorphic to $\mathcal{O}_H$.

If H is an infinite dimensional Hilbert space then $C^*(H)$, the C^*-algebra generated by H is canonically embedded in both $\mathcal{O}_H$ and $\mathscr{E}_H$. If H is in addition separable then $C^*(H)$ is the C^*-algebra $\mathcal{O}_\infty$ of Cuntz.

Let K be another Hilbert space in M. Assume that H and K either both have support I or both support strictly less than I. As in the case where H and K are finite dimensional, every unitary operator U from H to K defines an isomorphism λ_U from $\mathcal{O}_H$ to $\mathcal{O}_K$ (or from $\mathscr{E}_H$ to $\mathscr{E}_K$) satisfying

$$\lambda_U(\psi) = U\psi, \qquad \psi \in H.$$

Clearly λ_U restricts to an isomorphism from $C^*(H)$ to $C^*(K)$. In particular on $\mathcal{O}_H$

$$\sigma_K = \lambda_{\vartheta_{H,K}},$$

where $\vartheta_{H,K} \in (HK, KH)$ denotes the unitary operator permuting H and K. If $H = K$ and no confusion arises we simply write ϑ in place of $\vartheta_{K,K}$.

Let V be a multiplicative unitary on a Hilbert space $K \in \mathscr{H}(M)$ and W a corepresentation of V on $H \in \mathscr{H}(M)$, that is W is an invertible element of (KH, KH) that satisfies the following pentagon equation on K^2H

$$V_{12} W_{13} W_{23} = W_{23} V_{12}. \tag{2.2}$$

Note that in our setting

$$V_{12} = V,\ W_{13} = \vartheta\sigma_K(W)\,\vartheta = \sigma_K(\vartheta_{H,K})\, W\sigma_K(\vartheta_{K,H}),\ W_{23} = \sigma_K(W).$$

A representation $W \in (HK, HK)$ of V is defined by an analogous pentagon equation on HK^2 [10]. Since W is a representation of V if and only if $\vartheta W^* \vartheta$ or $\vartheta W^{-1} \vartheta$ is a corepresentation of the dual multiplicative unitary $\vartheta V^* \vartheta$, it will be sufficient to restrict our attention to the case of corepresentations of V.

In the following we shall need unbounded corepresentations of V. When such corepresentations are needed, we implicitly assume the existence of a fixed dense subspace K_0 of K such that V is bijective on the algebraic tensor product $K_0 \odot K_0$. We call an unbounded closed linear operator W densely defined on KH an unbounded corepresentation of V on H if for some dense subspace $H_W \subseteq H$ linearly spanned by an orthonormal system of H, $K_0 \odot H_W$ is a core for W contained also in the domain of W^*, W and W^* are bijections of $K_0 \odot H_W$ and (2.2) holds on $K_0 \odot K_0 \odot H_W$. Note that W is necessarily injective with dense range. One defines similarly an unbounded representation of V and one has again that W is an unbounded corepresentation of V on H if and only if $\vartheta W^{-1} \vartheta$ or the closure of $\vartheta W^* \vartheta \restriction_{H_W \odot K_0}$ is an unbounded representation of $\vartheta V^* \vartheta$ on H.

Note that if W is a corepresentation of V, possibly unbounded, then also W^{-1*} (or its closure on $K_0 \odot H_W$ in the unbounded case) is.

Let W and W' be two bounded corepresentations on H and H' respectively. We recall that the set of intertwiners between W and W' is

$$(W, W') = \{T \in (H, H') : \sigma_K(T)\, W = W' \sigma_K(T)\}.$$

In particular W and W' are called equivalent if (W, W') contains an invertible element, and are called unitarily equivalent if it contains a unitary operator. We denote by $\mathscr{E}(W, W')$ the set of all invertible elements of (W, W'). Clearly two unitary equivalent corepresentations W and W' are unitarily equivalent via the polar part of some invertible intertwiner.

If $E \in (W, W)$ is an orthogonal projection, the subcorepresentation of W induced by E is defined by restricting W to $KE(H)$.

Similarly, if W and W' are unbounded corepresentations, the set of intertwiners from W to W', denoted again by (W, W') consists of those preclosed operators T: $\mathscr{D}(T) = H_W \subseteq H \to H'$ such that $\mathrm{Ran}(T^* \restriction_{H_{W'}}) \subseteq H_W$, $\mathrm{Ran}(T) \subseteq H_{W'} \subseteq \mathscr{D}(T^*)$ and $I \times TW = W' I \times T$ on $K_0 \odot H_W$. Clearly $T \in (W, W')$ if and only if $T^* \in (W'^{-1*}, W^{-1*})$. In this setting the subset of equivalences $\mathscr{E}(W, W') \subset (W, W')$ consists of those intertwiners T such that T and $T^* \restriction H_{W'}$ are bijections between H_W and $H_{W'}$. If W and W' are unbounded representations we give analogous notions of intertwiners and equivalences.

Let E be an orthogonal projection on H such that $E(H_W) = H_1$ is a subspace of H_W linearly spanned by an orthonormal system. If $E \restriction_{H_W} \in (W, W)$, then the closure of the restriction of W to $K_0 \odot H_1$ defines a (possibly unbounded) subcorepresentation of W. We note explicitly that a W-stable subspace of the form $E(H_W) = H_1 \subseteq H_W$, that is $WK_0 \odot H_1 = K_0 \odot H_1$, is not a subcorepresentation in general. One needs also $W^* K_0 \odot H_1 = K_0 \odot H_1$.

If W and W' are (unbounded) corepresentations, we write $W \prec W'$ if W is equivalent to a subcorepresenation of W'.

Let $\{H_\alpha, \alpha \in A\}$ be a family of pairwise orthogonal Hilbert spaces in M, that is $H_\alpha^* H_\beta = 0$ if $\alpha \neq \beta$, and denote by $H = \bigoplus_{\alpha \in A} H_\alpha$ the norm closed linear subspace of M generated by these subspaces. If W_α is a corepresentation of V on H_α, we define the direct sum $W = \bigoplus_{\alpha \in A} W_\alpha$ as the closure of the linear operator on H which acts as W_α on KH_α. This is a bounded corepresentation of V iff $\sup_{\alpha \in A} \|W_\alpha\|$ and $\sup_{\alpha \in A} \|W_\alpha^{-1}\|$ are finite. If every W_α is an unbounded corepresentation, then W is again an unbounded corepresentation of V on H.

Since we work over a strict tensor category of Hilbert spaces, the category $\mathscr{C}(V)$ of bounded corepresentations $W \in M$ of V has a natural structure of strict tensor category, the strict tensor product is defined on the objects by

$$W \times W' = W_{12} W'_{13},$$

which is a corepresentation on HH', and on the arrows by

$$T \times T' = T\sigma_H(T') \in (W \times W', Z \times Z'), \qquad T \in (W, Z), \quad T' \in (W', Z').$$

The identity object is the identity operator on K. The full subcategory $\mathscr{T}(V)$ of unitary corepresentations is a strict tensor W^*-category.

Note that there is a faithful tensor *-functor from $\mathscr{T}(V)$ to $\mathscr{H}(M)$ given by $W \mapsto H$ and the identity on arrows. Since there is also a faithful *-functor from $\mathscr{H}(M)$ to $\mathrm{End}(M)$, the W^*-tensor category of endomorphisms of M, given by $H \to \sigma_H$ and the identity on the arrows, one can also regard $\mathscr{T}(V)$ as an endomorphism category of M.

If W and W' are unbounded corepresentations on H and H' we define the tensor product $W \times W'$ of W and W' as the closure of the restriction of $W_{12} W'_{13}$ to $K_0 \odot H_W \odot H_{W'}$.

Let $M(\mathscr{B})$ denote the multiplier algebra of some C^*-algebra $\mathscr{B}$. A homomorphism of C^*-algebras $\pi\colon \mathscr{A} \to M(\mathscr{B})$ is called non degenerate if for any approximate unit $\{u_\alpha\}$ of $\mathscr{A}$, $\pi(u_\alpha)$ converges to the identity operator I in the strict topology. In this case π extends to a strictly continuous *-homomorphism $\pi\colon M(\mathscr{A}) \to M(\mathscr{B})$ [15].

Following [10], by a Hopf C^*-algebra we mean a pair $(\mathscr{A}, \delta)$ constituted by a C^*-algebra $\mathscr{A}$ equipped with a non degenerate *-homomorphism $\delta\colon \mathscr{A} \to M(\mathscr{A} \otimes \mathscr{A})$, called the coproduct, satisfying $\delta(\mathscr{A})(I \otimes \mathscr{A} + \mathscr{A} \otimes I) \subseteq \mathscr{A} \otimes \mathscr{A}$, with I an adjoined unit to $\mathscr{A}$, and $id_{\mathscr{A}} \otimes \delta \circ \delta = \delta \otimes id_{\mathscr{A}} \circ \delta$, with $id_{\mathscr{A}}$ the identity map on $\mathscr{A}$.

A coaction of a Hopf C^*-algebra $(\mathscr{A}, \delta)$ on a unital C^*-algebra $\mathscr{B}$ is defined by a unital *-homomorphism $\lambda\colon \mathscr{B} \to M(\mathscr{A} \otimes \mathscr{B})$ such that $\delta \otimes id_{\mathscr{B}} \circ \lambda = id_{\mathscr{A}} \otimes \lambda \circ \lambda$.

3. The Conjugate Corepresentation

Our first aim is to give a notion of conjugate object of a corepresentation W of a multiplicative unitary V. We show that any such W has a conjugate provided the regular corepresentation V has a conjugate.

For any Hilbert space H in M we denote by J_H: $H \in \mathscr{H}(M) \to \bar{H} \in \mathscr{H}(M)$ the canonical antiunitary operator on the conjugate Hilbert space $\bar{H}$.

Let W be a bounded corepresentation of V on H. A corepresentation $\bar{W}$ of V on $\bar{H}$ is called the conjugate of W if it satisfies the conjugate property

$$(\psi\varphi, W\psi'\varphi') = (\bar{W}\psi\overline{\varphi'}, \psi'\bar{\varphi}), \tag{3.1}$$

for $\psi, \psi' \in K$, $\varphi, \varphi' \in H$, where we write for short $\bar{\varphi}$ on place of $J_H\varphi$.

Let t denote the transposition map from (H, H) to $(\bar{H}, \bar{H})$, $t(A) = J_H A^* J_H^*$. If H is finite dimensional then for every $K \in \mathscr{H}(M)$ the tensor product $id \times t$ extends to a linear map from (KH, KH) to $(K\bar{H}, K\bar{H})$. Then $\bar{W} = id \times t(W^*)$ defines a conjugate of W provided is invertible.

Let $T \in (H, H')$ be a linear operator and set $T^c = J_{H'} T J_H^*$. Then clearly $T \in (W, W')$ if and only if $T^c \in (\bar{W}, \overline{W'})$.

An unbounded conjugate $\bar{W}$ of W is an unbounded corepresentation of V on $\bar{H}$ such that, writing

$$H_{\bar{W}} = \overline{H_W},$$

then $WK_0 \odot H_W = W^*K_0 \odot H_W = K_0 \odot H_W$ and (3.1) holds for vectors $\psi, \psi' \in K_0$, $\varphi, \varphi' \in H_W$. If W itself is unbounded an unbounded conjugate of W is similarly defined with the only difference that the above identity is now required. In any case $\bar{W}$ admits W as a conjugate. In particular if $\bar{V}$ is an unbounded conjugate for the regular corepresentation V we require $H_{\bar{V}} = J_K(L_0)$. A suitably modified statement on intertwiners holds in the unbounded case.

It is easy to show that either in the unbounded and in the bounded case the corepresentation equation (2.2) for $\bar{W}$ follows in fact from (3.1).

An analogous definition of (possibly unbounded) conjugate representation goes through and clearly $\bar{W}$ is a conjugate corepresentation of W if and only if $\vartheta\bar{W}^*\vartheta$ is a conjugate representation of $\vartheta V^*\vartheta$.

Note that V admits a (unbounded) conjugate corepresentation $\bar{V}$ if and only if it admits a (unbounded) conjugate representation. It is given by $\overline{V^c}^*$.

We are interested in investigating on bounded corepresentations gifted with possibly unbounded conjugates since there are interesting cases, typically arising from compact quantum groups, where the regular corepresentation has an unbounded conjugate equivalent to V itself (cf. Example 4.4). This is the motivation for the following

Assumption. In the statements of this section all corepresentations in input without overbars are assumed to be bounded and their conjugates are assumed to be either all bounded or all unbounded.

It is easy to show that analogous statements hold in the case where the starting corepresentations are unbounded. In the following proposition we establish some basic properties of conjugate corepresentations. The proof can be deduced by direct computations.

3.1. PROPOSITION. *If W, W' and W_α, $\alpha \in A$ are corepresentations of V with conjugates then*

(a) *if $Z \prec W$ then there is $\bar{Z}$ and $\bar{Z} \prec \bar{W}$;*

(b) *$\bigoplus_{\alpha \in A} \bar{W}_\alpha$ is the conjugate of $\bigoplus_{\alpha \in A} W_\alpha$ whenever both these direct sums are well defined in M.*

Thus the class of corepresentations of V which have a conjugate object is closed under equivalence, subcorepresentations and direct sums. In the following proposition we prove that it is also closed under tensor products.

3.2. PROPOSITION. *Let W and W' be corepresentations of V acting on H and H' respectively with conjugates, then $I_K \times \vartheta_{\overline{H'}, \bar{H}} \circ \overline{W'} \times \bar{W} \circ I_K \times \vartheta_{\bar{H}, \overline{H'}}$ defines a conjugate of $W \times W'$.*

Proof. The operator $I_K \times \vartheta_{\overline{H'}, \bar{H}} \circ \overline{W'} \times \bar{W} \circ I_K \times \vartheta_{\bar{H}, \overline{H'}} = \overline{W'}_{13} \bar{W}_{12}$ is clearly a corepresentation of V, thus we have to show that for $\psi, \psi' \in K$, $\varphi, \varphi' \in H$ and $\eta, \eta' \in H'$

$$(\psi\varphi\eta, W_{12} W'_{13} \psi'\varphi'\eta') = (\overline{W'}_{13} \bar{W}_{12} \psi \overline{\varphi'}\, \overline{\eta'}, \psi' \bar{\varphi}\bar{\eta}).$$

Let $\{\psi_i\ i \in I\}$ and $\{\varphi_j\ j \in J\}$ be orthonormal bases of K and H respetively. In the unbounded case we may assume that these bases span linearly K_0 and H_W. Then the right hand side is

$$\sum_{i,j} (\bar{W}\psi\overline{\varphi'}, \psi_i \bar{\varphi}_j)(\overline{W'}_{13} \psi_i \bar{\varphi}_j \overline{\eta'}, \psi' \bar{\varphi}\bar{\eta})$$
$$= \sum_{i,j} (\psi\varphi_j, W\psi_i\varphi')(\psi_i\varphi\eta, W'_{13}\psi'\varphi_j\eta').$$

Now

$$\begin{aligned}\sum_j (\psi\varphi_j, W\psi_i\varphi')\, W'_{13}\psi'\varphi_j\eta' &= \sum_j (\varphi_j, \psi^* W\psi_i\varphi')\, W'_{13}\psi'\varphi_j\eta' \\ &= W'_{13}\psi'(\psi^* W\psi_i\varphi')\,\eta' \\ &= \sigma_K(\psi^* W\psi_i\varphi')\, W'\psi'\eta',\end{aligned}$$

where, in the unbounded case, we check the above identities in the weak topology of KHH', so the previous sum is also equal to

$$\begin{aligned}\sum_i (\psi_i\varphi\eta, \sigma_K(\psi^* W\psi_i\varphi')\, W'\psi'\eta') &= (\varphi\eta, \psi^* W\sigma_K(\varphi')\, W'\psi'\eta') \\ &= (\psi\varphi\eta, W_{12} W'_{13}\psi'\varphi'\eta'). \quad \blacksquare\end{aligned}$$

In view of the last proposition we see that the conjugates of W' and $W \times W'$ actually determine that of W. This suggests the following result.

3.3. PROPOSITION. *Let W and W' be corepresentations of V such that $W \times W'$ admits a conjugate. Then W has a conjugate corepresentation if and only if W' does.*

Proof. We first prove the proposition in the bounded case. We assume that W' has the conjugate and prove the existence of the conjugate of W. The remaining statement is proved similarly. We show that the operator $Z = \overline{W'}_{13}{}^{-1}\overline{W \times W'}$ acts as the identity operator on the third Hilbert space $\overline{H'}$ and that the invertible operator $\overline{W}$ defined by Z on $K\overline{H}$ is the desired conjugate corepresentation of W. We show that for $\psi, \psi' \in K$, $\varphi, \varphi' \in H$, $\eta, \eta' \in H'$,

$$(\psi\varphi, W\psi'\varphi')(\eta, \eta') = (Z\psi\overline{\varphi'}\,\overline{\eta'}, \psi'\bar{\varphi}\bar{\eta}). \tag{3.2}$$

Let $\{\psi_i, i \in I\}$ and $\{\eta_j, j \in J\}$ be orthonormal bases of K and H' respectively. The right hand side of (3.2) is

$$\sum_{i,j} (\overline{W \times W'}\psi\overline{\varphi'}\,\overline{\eta'}, \psi_i\bar{\varphi}\overline{\eta_j})(\psi_i\overline{\eta_j}, \overline{W'}^{-1*}\psi'\bar{\eta})$$

$$= \sum_{i,j} (\psi\varphi\eta_j, W_{12}W'_{13}\psi_i\varphi'\eta')(\psi_i\overline{\eta_j}, \overline{W'}^{-1*}\psi'\bar{\eta}). \tag{3.3}$$

Now $\sigma_K(\overline{\eta_j}^*)\,\overline{W'}^{-1*}\psi'\bar{\eta} \in K$ so

$$\sum_i (\psi_i\overline{\eta_j}, \overline{W'}^{-1*}\psi'\bar{\eta})\,\psi_i = \sigma_K(\overline{\eta_j}^*)\,\overline{W'}^{-1*}\psi'\bar{\eta}$$

hence (3.3) is also equal to

$$\sum_j (\psi\varphi\eta_j, WW'_{13}\sigma_K(\overline{\eta_j}^*)\,\overline{W'}^{-1*}\psi'\bar{\eta}\varphi'\eta')$$

so the proof of (3.2) will be complete if one proves that

$$\sum_j \sigma_{KH'}(\eta_j^*)\,W'_{13}\sigma_K(\overline{\eta_j}^*)\,\overline{W'}^{-1*}\psi'\bar{\eta}\varphi'\eta' = \psi'\varphi'(\eta, \eta')$$

in the weak topology of KH. The proof of this fact is a simple consequence of the existence of the conjugate of W' and goes in the same spirit of the above computations so we decide to omit it. The fact that $\overline{W}$ satisfies the pentagon equation (2.2) follows from the fact that it satisfies the conjugate property, however it can also be proved directly. In terms of Z we have to show that

$$V_{12}Z_{134}Z_{234} = Z_{234}V_{12}.$$

Note that $\overline{W'}_{24}^{-1}$ commutes with Z_{134} since the latter acts as the identity operator on the second and the fourth factor, so the left hand side is

$$\begin{aligned} V_{12}\overline{W'}_{24}^{-1}\overline{W'}_{14}^{-1}\overline{W\times W'}_{134}\overline{W\times W'}_{234} &= \overline{W'}_{24}^{-1}V_{12}\overline{W\times W'}_{134}\overline{W\times W'}_{234} \\ &= \overline{W'}_{24}^{-1}\overline{W\times W'}_{234}V_{12}=Z_{234}V_{12}. \end{aligned}$$

If W and W' are unbounded corepresentations on H_W and $H_{W'}$ respectively then we may again define the operator $Z=\overline{W'}_{13}^{-1}\overline{W\times W'}$ on $K_0\odot \overline{H_W}\odot\overline{H_{W'}}$. Clearly Z and Z^* are bijective on this subspace. The same computations in the bounded case with suitable choices of orthonormal bases show that Z defines a preclosed linear operator on $K_0\odot\overline{H_W}$ whose closure is a conjugate of W. ∎

3.4. COROLLARY. *Let V be the regular corepresentation gifted with a conjugate $\bar V$ and let W be a corepresentation of V on H. Assume that either $\bar V$ is bounded and invertible or that there is a dense subspace $H_W\subset H$ such $WK_0\odot H_W=W^*K_0\odot H_W=K_0\odot H_W$. Then W admits a conjugate $\bar W$, defined by*

$$\bar W_{13}\bar V_{12}=I\times W^c\circ\bar V_{12}\circ I\times W^{-1^c}.$$

In particular if $\bar V$ is isometric then $\bar W$ is isometric if and only if W is isometric.

Proof. The pentagon equation simply means that the corepresentations $V\times W$ and $V\times I$, with I the trivial corepresentation of V are equivalent via W, so the conclusion follows from the previous proposition. ∎

Let W be a bounded linear operator on KH. Following S. Baaj and G. Skandalis we define for every $\omega\in(K,K)_*$, the predual of (K,K), the operator $L^W(\omega)\in(H,H)$

$$(\varphi, L^W(\omega)\,\varphi')=\omega(\sigma_K(\varphi^*)\,W\sigma_K(\varphi')),\qquad \varphi,\varphi'\in H,$$

in particular if $\omega=\omega_{\psi,\psi'}$, $\psi,\psi'\in K$ then $L^W(\omega)=\psi^*W\psi$, and we associate with every multiplicative unitary V two Banach subspaces of (K,K), $\mathscr{A}(V)$ and $\hat{\mathscr{A}}(V)$, defined respectively as the norm closures of

$$\mathscr{A}_0(V)=\{L^V(\omega),\omega\in(K,K)_*\},$$
$$\hat{\mathscr{A}}_0(V)=\{\rho_V(\omega):=L^{\vartheta V^*\vartheta}(\omega)^*,\omega\in(K,K)_*\}=\mathscr{A}_0(\vartheta V^*\vartheta)^*.$$

In the case where we need to use unbounded corepresentations it is useful to define $\mathscr{A}_0(V)$ and $\hat{\mathscr{A}}_0(V)$ as the linear spaces generated by elements of the above form, with $\omega = \omega_{\psi, \varphi}$, where ψ, φ are elements of the fixed dense subspace K_0. The fundamental property of the regular corepresentation implies that both $\mathscr{A}(V)$ and $\hat{\mathscr{A}}(V)$ are Banach algebras. In [10] Baaj and Skandalis introduce a sufficient condition on V, called regularity, for $\mathscr{A}(V)$ and $\hat{\mathscr{A}}(V)$ to be Hopf C^*-algebras. Coproducts are then defined by

$$\delta(X) = VXV^*, \qquad X \in \mathscr{A}(V),$$

$$\hat{\delta}(X) = \vartheta V^* \sigma_K(X)\, V\vartheta, \qquad X \in \hat{\mathscr{A}}(V).$$

The coinverse $\kappa: \mathscr{A}_0(V) \to \mathscr{A}(V)$ (resp. $\hat{\kappa}: \hat{\mathscr{A}}_0(V) \to \hat{\mathscr{A}}(V)$) is a linear antihomomorphism of algebras defined by $\kappa(L^V(\omega)) = L^V(\omega^*)^* = L^{V^*}(\omega)$ (resp. $\hat{\kappa}(\rho_V(\omega)^*) = \rho_V(\omega^*) = \rho_{V^*}(\omega)^*$). So the composition of the coinverse with the *-operation has square the identity map on both $\mathscr{A}_0(V)$ and $\hat{\mathscr{A}}_0(V)$.

If we associate to a locally compact group the multiplicative unitary

$$(Vf)(g, h) = f(g, hg^{-1}), \qquad f \in L^2(G \times G), \quad g, h \in G \tag{3.4}$$

then $\hat{\mathscr{A}}(V) = \mathscr{C}_0(G)$. In this case the selfadjointness of $\hat{\mathscr{A}}(V)$ can be regarded as a consequence of the fact that the regular representation of G is selfconjugate. Now the notion of selfconjugate regular corepresentation appears also in [10] in the special case where $\bar{V}$ is unitary. It is shown there that any such V is then regular. The same arguments generalize to show the following

3.5. Theorem. *Any multiplicative unitary V which admits a (possibly unbounded) conjugate representation or corepresentation, equivalently, is regular.*

Proof. To fix ideas we assume that V admits an unbounded conjugate corepresentation $\bar{V}$. Let $\omega \in (K, K)_*$ and let $T \in L^1(K) \subseteq L^2(K)$ be a trace class operator on K inducing ω, that is $\omega(X) = tr(TX)$. We regard T as an element of $L^2(K) = \bar{K}K$. Our aim is to show that $(id \times \omega)(\vartheta V) = \bar{V}^c T$ as an element of $L^2(K) \subseteq (K, K)$ if $\omega = \omega_{\varphi, \varphi'}$, $\varphi, \varphi' \in K_0$. Since $\bar{V}^c$ has dense range, the regularity of V would therefore follow. We have to show that for vectors $\psi, \psi' \in K_0$, $(\psi', (id \times \omega_{\varphi, \varphi'})(\vartheta V)\, \psi) = (\bar{\psi}\psi', \bar{V}^c \bar{\varphi}\varphi')$. Now the left hand side is

$$(\psi'\varphi, \vartheta V\psi\varphi') = (\varphi\psi', V\psi\varphi') = (\bar{V}\varphi\overline{\varphi'}, \psi\overline{\psi'}) = (\bar{\psi}\psi', \bar{V}^c\bar{\varphi}\varphi'). \quad \blacksquare$$

We now want to link the selfconjugateness of the multiplicative unitary V with the coinverse map defined on the associated $\hat{\mathscr{A}}_0(V)$ and $\mathscr{A}_0(V)$.

3.6. PROPOSITION. *Let W be a corepresentation of V on H. If $S \in (H, \bar{H})$ is an invertible element then the following are equivalent*:

(a) *there is a bounded conjugate corepresentation $\bar{W}$ and $S \in \mathscr{E}(W, \bar{W})$*;

(b) $L^W(\omega)^c\, S = SL^W(\omega^*)$, $\omega \in (K, K)_*$.

If W is unitary then $\bar{W}$ is unitary and equivalent to W if and only if there is a unitary S.

If $S: H_W \subset H \to \overline{H_W} \subseteq \bar{H}$ is a preclosed densely defined linear bijection such that S^* admits $\overline{H_W}$ as a core and is a bijection from $\bar{H}_W$ to H_W, then the following are equivalent:

(a′) *there is an unbounded conjugate $\bar{W}$ of W and $S \in \mathscr{E}(W, \bar{W})$*;

(b′) $L^W(\omega_{\psi', \psi})^c\, S \subseteq \bar{S} L^W(\omega_{\psi, \psi'})$ $\psi, \psi' \in K_0$, *with $\bar{S}$ the closure of S.*

Proof. We prove only the second statement, the first can be proved similarly. (a′) is clearly equivalent to

$$(\psi\varphi,\, W\psi'\varphi') = (W\psi S^{-1}\overline{\varphi'},\, \psi' S^* \bar{\varphi}), \qquad \psi, \psi' \in K_0, \quad \varphi, \varphi' \in H_W.$$

Now this equation can be rewritten as

$$(L(\omega_{\psi, \psi'})^c\, \overline{\varphi'}, \bar{\varphi}) = L(\omega_{\psi', \psi})\, S^{-1}\overline{\varphi'}, S^* \bar{\varphi}).$$

This shows that $L(\omega_{\psi', \psi})\, S^{-1}\overline{\psi'}$ is in the domain of $S^{**} = \bar{S}$ and the proof is complete. ∎

This proposition implies

3.7. COROLLARY. *Let V be a regular multiplicative unitary on K^2. If $T \in (K, K)$ is an antilinear invertible element then the following are equivalent*:

(a) *V admits a bounded conjugate corepresentation (resp. representation) $\bar{V}$ and $S = J_K T^* \in \mathscr{E}(V, \bar{V})$*;

(b) $\kappa(a) = Ta^* T^{-1}$, $a \in \mathscr{A}_0(V)$ (*resp.* $\hat{\kappa}(a) = T^{-1} a^* T$, $a \in \hat{\mathscr{A}}_0(V)$).

If T is a closed antilinear operator such that T and T^* admit K_0 as a core and are bijections on it then the following are equivalent:

(a′) *V admits an unbounded conjugate corepresentation (resp. representation) $\bar{V}$ and $S = J_K T^* \in \mathscr{E}(V, \bar{V})$*;

(b′) $\kappa(a)\, T \subseteq Ta^*$, $a \in \mathscr{A}_0(V)$ (*resp.* $T\hat{\kappa}(a) \subseteq a^* T$, $a \in \hat{\mathscr{A}}(V)$).

Let $\bar{V}$ be a (possibly unbounded) conjugate representation of V equivalent to V via S. Then $\omega_{\psi, \varphi} \times id(V^*) = \omega_{T^*\varphi, T^{-1}\psi}(V)$, for vectors ψ, φ in K (or in K_0), with $T = J_K^* S$. This shows immediately that $\mathscr{A}_0(V)$ is *-invariant and that κ is an antiautomorphism on $\mathscr{A}_0(V)$. If in particular V is self-conjugate also as a corepresentation we deduce by the above corollary, using polar decomposition of closed operators, that $\kappa = \kappa^{-1}$ on $\mathscr{A}_0(V)$ (or, equivalently, $\hat{\kappa} = \hat{\kappa}^{-1}$ on $\hat{\mathscr{A}}_0(V)$), if and only if one of the conjugates (and hence both), is isometric. In this case the coinverses are implemented by antiunitary operators.

Let $(\mathscr{A}, \delta)$ be a Hopf C^*-algebra in the sense of [15] or a Hopf–von Neumann algebra in the sense of [16; Definition 1.2.1]. Recall that a coinvolution on $(\mathscr{A}, \delta)$ is a linear involutive $*$-antiautomorphism κ of $\mathscr{A}$. Then $(\mathscr{A}, \delta)$ is a coinvolutive Hopf C^*-algebra, or Hopf–von Neumann algebra if

$$\delta \circ \kappa = \Theta \circ \kappa \otimes \kappa \circ \delta, \tag{3.5}$$

where Θ is the flip automorphism of $\mathscr{A} \otimes \mathscr{A}$.

We note the following consequence of the above corollary.

3.8. THEOREM. *Let V be a multiplicative unitary endowed with a conjugate corepresentation $\bar{V}$ unitarily equivalent to V itself. Then κ extends to a coinvolution on $(\mathscr{A}(V), \delta)$ making it into a coinvolutive Hopf C^*-algebra. Moreover δ and κ extend to $\mathscr{A}(V)''$ making it into a coinvolutive Hopf–von Neumann algebra.*

Proof. Clearly δ and κ extend to normal maps on $\mathscr{A}(V)''$ so it suffices to show that (3.5) holds on elements of the form $\psi^* V\varphi$, $\psi, \varphi \in K$. Now $\kappa(\psi^* V\varphi) = \psi^* V^* \varphi$, so if $\{\psi_i,\ i \in I\}$ is an orhonormal basis of K,

$$\begin{aligned}
\delta \circ \kappa(\psi^* V\varphi) &= V\psi^* V^* \varphi V^* = \psi^* V_{23} V_{12}^* V_{23}^* \varphi \\
&= \psi^* V_{13}^* V_{12}^* \varphi = \sum_i \sigma_K(\psi^* V^* \psi_i)\, \psi_i^* V^* \varphi \\
&= \sum_i \Theta(\psi^* V^* \psi_i \sigma_K(\psi_i^* V^* \varphi)) = \Theta \circ \kappa \otimes \kappa(\psi^* V V_{13} \varphi) \\
&= \Theta \circ \kappa \otimes \kappa(\psi^* V_{23} V_{12} V_{23}^* \varphi) = \Theta \circ \kappa \otimes \kappa \circ \delta(\psi^* V\varphi). \quad \blacksquare
\end{aligned}$$

Conversely, regular corepresentations associated to Kac–von Neumann algebras [16; Chapter 2] have unitary conjugates (cf. also Example 4.2).

4. EXAMPLES OF SELF-CONJUGATE REGULAR REPRESENTATIONS

4.1. Let G be a locally compact group with V defined as in (3.4). The notion of conjugate corepresentation reduces to the ordinary definition of conjugate representation of G and V corresponds to the regular representation which is selfconjugate. The operator S of proposition 3.6 relative to V can be chosen to be the complex conjugation of $L^2(G)$ which is clearly antiunitary.

4.2. Let V be the adjoint of the fundamental unitary associated with a Kac–von Neumann algebra as in [16; Chapter 2]. Then as in the group case the regular corepresentation V admits a unitary conjugate $\bar{V}$ which is unitarily equivalent to V by [10; Example 3.4.3].

4.3. Let $(\mathcal{A}, \delta)$ be a unital Hopf C^*-algebra satisfying $\delta(\mathcal{A})\, I \otimes \mathcal{A} = \mathcal{A} \otimes \mathcal{A}$ endowed a right Haar measure φ which is also a trace. If (K, π, ξ) is the corresponding GNS construction let V be the multiplicative unitary associated with $(\mathcal{A}, \delta)$ as in [10; Example 1.2.1]. Then the antiunitary operator j of K defined by $j\pi(x)\,\xi = \pi(x^*)\,\xi$ makes V selfconjugate as a representation.

Regular corepresentations gifted with unitary conjugate objects reflect the property that the natural coinverse of the underlying Hopf C^*-algebra is involutive, as the following example shows.

4.4. Let $(\mathcal{A}, \delta)$ be a Woronowicz C^*-algebra in the sense of [10; Définition 4.1]. This means that there is a subset $\{u^p_{ij}, p \in \mathbb{N}, i, j = 1, ..., n_p\}$ such that for any p the matrix $u^p = (u^p_{ij}) \in M_{n_p}(\mathcal{A})$ is unitary, its entries generate *a* $*$-algebra $\mathcal{B}_p$ such that $\mathcal{B}_p \subseteq \mathcal{B}_{p+1}$ and the union $\mathcal{B} = \bigcup_{p \in \mathbb{N}} \mathcal{B}_p$ is a dense $*$-subalgebra of $\mathcal{A}$ endowed with a linear antiautomorphism κ, the coinverse, satisfying for any $p \in \mathbb{N}$

(a) $\kappa(b)^* = \kappa^{-1}(b^*)$, $b \in \mathcal{B}$;

(b) $\delta(u^p_{ij}) = \sum_{k=1}^{n_p} u^p_{ik} \otimes u^p_{kj}$;

(c) $\kappa(u^p_{ij}) = u^{p*}_{ji}$.

Now on $\mathcal{B}$ the coinverse satisfies

$$\delta \circ \kappa = \Theta \circ \kappa \otimes \kappa \circ \delta, \tag{4.1}$$

$\mathcal{B}$ is also endowed with a $*$-character e, the counit, such that $e(u^p_{ij}) = \delta_{ij}$, and

$$m \circ \kappa \otimes id \circ \delta(b) = m \circ id \otimes \kappa \circ \delta(b) = e(b)\, I$$

for $b \in \mathcal{B}$, where $m\colon \mathcal{B} \odot \mathcal{B} \to \mathcal{B}$ denotes the multiplication map.

Let φ denote the unique Haar measure of $\mathcal{A}$ (faithful on $\mathcal{B}$), (π, K, ξ) the corresponding GNS construction and V the operator defined on the algebraic tensor product $\pi(\mathcal{B})\,\xi \odot \pi(\mathcal{B})\,\xi$ by

$$V\pi(a)\,\xi \otimes \pi(b)\,\xi = \pi \otimes \pi \circ \kappa^{-1} \otimes id \circ \delta(b)(\pi(a)\,\xi \otimes \xi). \tag{4.2}$$

Then V extends to a regular multiplicative unitary on $K \otimes K$ with cofixed points $\mathbb{C}\xi$ [10], [17], [18]. Regarded as a corepresentation, we call V the left regular corepresentation.

The Hopf C^*-algebra $\hat{\mathscr{A}}(V)$ coincides with $\pi(\mathscr{A})$ and is a Woronowicz C^*-algebra in a natural way [10; Théorème 4.2], hence, up to replacing $\mathscr{A}$ with its image via π, we assume π faithful.

Similarly, the operator

$$W\pi(a)\,\xi\otimes\pi(b)\,\xi=\pi\otimes\pi\circ\delta(a)(\xi\otimes\pi(b)\,\xi)$$

extends to a regular multiplicative unitary on $K\otimes K$ with fixed points $\mathbb{C}\xi$.

Direct computations show that $\vartheta W\vartheta$ is a corepresentation of V, that we call the right regular corepresentation.

Note that V and $\vartheta W\vartheta$ are bijective on $\pi(\mathscr{B})\,\xi\odot\pi(\mathscr{B})\,\xi$ and that the invertible densely defined operator K^0: $\pi(\mathscr{B})\,\xi\subseteq K\to K$, $\pi(\mathscr{B})\,\xi\to\pi\circ\kappa(b)\,\xi$ intertwines $\vartheta W\vartheta$ with V on that subspace.

By [10; Proposition 4.8] or [19; Théorème 7.2.2], $\kappa=\kappa^{-1}$ if and only if φ is a trace. In this case K^0 extends to a unitary intertwiner. More generally K^0 is preclosed and the unitary operator X coming from the polar decomposition of its closure K intertwines again the two regular corepresentations [10; Proposition 5.2]. Similarly, $\vartheta V\vartheta$ is a representation of W and X intertwines W with $\vartheta V\vartheta$.

A unitary corepresentation Z of V (or a unitary representation of W) on a finite dimensional Hilbert space H has the form $Z=\sum_{h,\,k=1}^{n} u_{hk}\otimes e_{hk}$, with n the dimension of H, e_{hk} a system of matrix units on H and $u=(u_{hk})$ a unitary element of $M_n(\mathscr{A})$ such that $\delta(u_{hk})=\sum_j u_{hj}\otimes u_{jk}$. If $u\in M_n(\mathscr{B})$ then Z has a conjugate corepresentation $\bar{Z}=\sum_{h,\,k} u^*_{hk}\otimes\overline{e_{hk}}$ [17; (1.38), (1.39)].

We introduce two densely defined antilinear operators on the common dense domain $\pi(\mathscr{B})\,\xi\subseteq K$,

$$T_0\pi(b)\,\xi=\pi(\kappa(b^*))\,\xi,\qquad G_0\pi(b)\,\xi=\pi(\kappa^{-1}(b^*))\,\xi=\pi(\kappa(b)^*)\,\xi.$$

To prove the following proposition we shall make use of the character group of $\mathscr{B}$, $z\in\mathbb{C}\to f_z$ defined in [17; Theorem 5.6]. Recall that if f and g are linear functionals on $\mathscr{B}$ and $b\in\mathscr{B}$ there are convolution products: $b*f=f\otimes id\circ\delta(b)\in\mathscr{B}$, $f*b=id\otimes f\circ\delta(b)\in\mathscr{B}$ and $f*g(b)=f\otimes g\circ\delta(b)$. With the last product the set of all linear functionals on $\mathscr{B}$ becomes a semigroup with identity the counit. We shall need the following properties of the semigroup f_z. For all $a, b\in\mathscr{B}$ and complex numbers z, z':

$$f_z*f_{z'}=f_{z+z'},\qquad f_0=e,\tag{4.3}$$

$$\text{the maps } a\to f_z*a,\ a\to a*f_z \text{ are automorphisms of } \mathscr{B},\tag{4.4}$$

$$(f_z*a)^*=f_{-\bar{z}}*a^*,\qquad (a*f_z)^*=a^**f_{-\bar{z}},\tag{4.5}$$

$$f_z*\kappa(a)=\kappa(a*f_{-z}),\qquad \kappa(a)*f_z=\kappa(f_{-z}*a),\tag{4.6}$$

$$\kappa^2(a) = f_{-1} * a * f_1, \tag{4.7}$$

$$\varphi(a * f_z) = \varphi(f_z * a) = \varphi(a), \tag{4.8}$$

$$\varphi(ab) = \varphi(bf_1 * a * f_1). \tag{4.9}$$

We then have

4.5. PROPOSITION. *The operators T_0 and G_0 are preclosed. Their closures, denoted respectively by T and G are injective, with dense range and satisfy $T = T^{-1}$, $G = G^{-1}$, $G = T^*$.*

Proof. The Haar measure φ is κ-invariant, hence for $b, b' \in \mathscr{B}$,

$$\begin{aligned}(T_0 \pi(b)\, \xi, \pi(b')\, \xi) &= \varphi(\kappa(b^*)^*\, b') = \varphi(\kappa^{-1}(b)\, b') = \varphi(\kappa(b')\, b) \\ &= \varphi((\kappa^{-1}(b'^*)^*\, b) = (G_0 \pi(b')\, \xi, \pi(b)\, \xi).\end{aligned}$$

This shows that T_0 and G_0 are preclosed and that their closures T and G satisfy $G \subseteq T^*$, $T \subset G^*$. Clearly T_0 and G_0 are involutive, hence so are T and G.

Let us define the operator U by $U\pi(a)\, \xi = \pi(f_{1/2} * \kappa(a^*) * f_{-1/2})\, \xi$ then for $a, b \in \mathscr{B}$,

$$(f_{1/2} * \kappa(a^*) * f_{-1/2})^* = f_{-1/2} * \kappa^{-1}(a) * f_{1/2} = f_{1/2} * \kappa(a) * f_{-1/2}$$

by (4.3) and (4.5), hence

$$\begin{aligned}&\varphi(f_{1/2} * \kappa(a^*) * f_{-1/2})^*\, f_{1/2} * \kappa(b^*) * f_{-1/2}) \\ &\quad = \varphi(f_{1/2} * \kappa(a) * f_{-1/2} f_{1/2} * \kappa(b^*) * f_{-1/2}) \\ &\quad = \varphi(f_{1/2} * \kappa(b^* a) * f_{-1/2}) = \varphi(b^* a),\end{aligned}$$

where we have used successively (4.4), (4.8) and the κ-invariance of φ. This shows that U extends to an antiunitary operator on K. The property $U^2 = I$ can be deduced easily. Note that for $\eta = \pi(a)\, \xi$, $UG\eta = \pi(f_{1/2} * \kappa^2(a) * f_{-1/2})\, \xi$ $= \pi(f_{-1/2} * a * f_{1/2})\, \xi$, hence

$$\begin{aligned}(\eta, UG\eta) &= \varphi(a^* f_{-1/2} * a * f_{1/2}) \\ &= \varphi(f_{-1/2} * (f_{1/4} * a^* * f_{-1/4} f_{-1/4} * a * f_{1/4}) * f_{1/4}) \\ &= \varphi((f_{-1/4} * a * f_{1/4})^*\, f_{-1/4} * a * f_{1/4}) \geqslant 0.\end{aligned}$$

Let $u = (u_{hk}) \in M_n(\mathscr{B})$ define a unitary finite dimensional corepresentation of V. Now G and its polar decomposition leave the linear span of $\{\pi(u_{hk})\, \xi,\ h, j = 1, ..., n\}$ globally invariant and $\mathscr{B}$ is linearly spanned by subspaces of this form [17; Proposition 4.7], so $G = U\,|G|$ and $G^* = |G|\; U$ admits $U\pi(\mathscr{B})\, \xi = \pi(\mathscr{B})\, \xi$ as a core. ∎

Note that if $\kappa=\kappa^{-1}$, T and G are selfadjoint antiunitary operators.

4.6. LEMMA. *The Haar measure φ satisfies for $a, b \in \mathscr{B}$,*

$$id \otimes \varphi(\delta(a)\, I \otimes b) = id \otimes \varphi(I \otimes a\kappa^{-1} \otimes id \circ \delta(b)), \tag{4.10}$$

$$\varphi \otimes id(a \otimes I\, \delta(b)) = \varphi \otimes id(id \otimes \kappa^{-1} \circ \delta(a)\, b \otimes I). \tag{4.11}$$

Proof. By (4.34) of [17] the linear map defined on $\pi(\mathscr{B})\,\xi \odot \pi(\mathscr{B})\,\xi$ by $\pi(a)\,\xi \otimes \pi(b)\,\xi \to \pi \otimes \pi(\delta(b)\, a \otimes I)$ is the inverse of V, and V is unitary hence, for $a, b, c \in \mathscr{B}$,

$$\begin{aligned}\varphi(c\; id \otimes \varphi(\delta(a)\, I \otimes b)) &= \varphi \otimes \varphi(c \otimes I\, \delta(a)\, I \otimes b)\\ &= (V^* \pi(c^*)\,\xi \otimes \pi(a^*)\,\xi,\, \xi \otimes \pi(b)\,\xi)\\ &= \varphi \otimes \varphi(c \otimes a\,\kappa^{-1} \otimes id \circ \delta(b))\\ &= \varphi(c\; id \otimes \varphi(I \otimes a\,\kappa^{-1} \otimes id \circ \delta(b))).\end{aligned}$$

Now φ is faithful on $\mathscr{B}$, so (4.10) follows. Again, by (4.36) of [17] the inverse of W on $\pi(\mathscr{B})\,\xi \odot \pi(\mathscr{B})\,\xi$ is $\pi(a)\,\xi \otimes \pi(b)\,\xi \to \pi \otimes \pi(id \otimes \kappa \circ \delta(a)\, I \otimes b)$ and (4.9) follows from analogous computations. ∎

We now come to the computation of the conjugates of V and W. Note that the antilinear operator $S_0 \colon \pi(b)\,\xi \to \pi(b^*)\,\xi$ defined on $\pi(\mathscr{B})\,\xi$ is bijective and preclosed since $S_0 \subseteq K^{-1}T$ implies $\pi(\mathscr{B})\,\xi \subseteq \mathscr{D}(S_0^*)$. Moreover the adjoint of S, the closure of S_0, is bijective on $\pi(\mathscr{B})\,\xi$. Let $\bar{K}$ denote the conjugate Hilbert space of K and $J \colon K \to \bar{K}$ the canonical antiunitary operator. Then we deduce the following result. I would like to thank the Referee for suggesting the first proof we present here below.

4.7. THEOREM. *The regular corepresentations* (*resp. representations*) *V and W admit* (*possibly unbounded*) *conjugates $\bar{V}$ and $\bar{W}$ and we have*

$$JTK^{-1} \in \mathscr{E}(V, \bar{V}), \qquad JG \in \mathscr{E}(W, \bar{W})$$

$$(\text{resp. } JG \in \mathscr{E}(V, \bar{V}), \qquad JS \in \mathscr{E}(W, \bar{W})).$$

Furthermore $\bar{V}$ and $\bar{W}$ extend to unitary operators if and only if κ is involutive.

1st *Proof.* We only give a proof of the part of the theorem concerning corepresentations, the remaining statement can be proved similarly. Corepresentations of V are representations of $\mathscr{A}(V) = \bigoplus M_{n_k}(\mathbb{C})$, where $\bigoplus$ denotes a c_0 direct sum [10]. Now the regular corepresentation is a faithful representation of $\mathscr{A}(V)$, therefore every irreducible is finite dimensional and appears in V with multiplicity equal to its dimension thanks to the orthogonality conditions of [17]. Now a direct sum of corepresentations having conjugates has itself a conjugate. This reduces the proof to the case of an irreducible corepresentation, in which case all the calculations can be found in [171. ∎

We also give a direct proof.

2nd *Proof.* The operators TK^{-1}, G and their adjoints are bijective on $\pi(\mathscr{B})\,\xi$, hence $V_1 = I\otimes JTK^{-1}V\,I\otimes KTJ^*$ and $W_1 = I\otimes JG\;W\,I\otimes GJ^*$ and their adjoints are bijective on $\pi(\mathscr{B})\,\xi \odot \overline{\pi(\mathscr{B})\,\xi}$. In particular V_1 and W_1 are preclosed and their closures are corepresentations of V and W respectively, possibly unbounded. To complete the proof we have to check that the pairs (V, V_1) and (W, W_1) satisfy the conjugate property. We start with (V, V_1). Now $K\in\mathscr{E}(\vartheta W\vartheta, V)$, so we have to show that for vectors $a, b, a', b'\in\mathscr{B}$

$$\begin{aligned}(WT\pi(b')\,\xi\otimes\pi(a)\,\xi,\, T^*\pi(b)\,\xi\otimes\pi(a')\,\xi)&\\ =(\pi(a)\,\xi\otimes\pi(b)\,\xi,\, V\pi(a')\,\xi\otimes\pi(b')\,\xi).&\qquad(4.12)\end{aligned}$$

Now the left hand side is

$$\begin{aligned}&\varphi\otimes\varphi(I\otimes a^*\delta\circ\kappa^{-1}(b')\,\kappa(b)^*\otimes a')\\ &\quad=\varphi(a^*\varphi\otimes id\circ\Theta\circ\kappa^{-1}\otimes\kappa^{-1}(I\otimes b^*\,\delta(b'))\,a')\\ &\quad=\varphi(a^*\,id\otimes\varphi(I\otimes b^*\kappa^{-1}\otimes id\circ\delta(b'))\,a'),\end{aligned}$$

since κ is an antihomomorphism and φ is κ-invariant, and this coincides with the right hand side of (4.12).

We now prove that W and W_1 satisfy the conjugate property, that is

$$\begin{aligned}(W\pi(a)\,\xi\otimes G\pi(b')\,\xi,\, \pi(a')\,\xi\otimes G^*\pi(b)\,\xi)&\\ =(\pi(a)\,\xi\otimes\pi(b)\,\xi,\, W\pi(a')\,\xi\otimes\pi(b')\,\xi).&\qquad(4.13)\end{aligned}$$

The left hand side coincides with

$$\begin{aligned}&\varphi(\kappa(b')\,\varphi\otimes id(\delta(a^*)\,a'\otimes I)\,\kappa(b^*))\\ &\quad=\varphi(b^*\varphi\otimes id(id\otimes\kappa^{-1}\circ\delta(a^*)\,a'\otimes I)\,b')\end{aligned}$$

by the κ-invariance of φ, and therefore with the right hand side of (4.13) by (4.11).

Finally, if κ is involutive then K, JT and JG are isometric, so $\bar{V}$ and $\bar{W}$ extend to unitary operators. Conversely, if one of $\bar{V}$ and $\bar{W}$, say $\bar{V}$, extends to a unitary corepresentation then any unitary corepresentation of V has a unitary conjugate by Corollary 3.4. In particular for any $p \in \mathbb{N}$ the matrix $\overline{u^p} = (u^{p*}_{hk}) \in M_{n_p}(\mathscr{B})$ must be unitary, so $\kappa(u^{p*}_{hk}) = u^p_{kh} = \kappa(u^p_{hk})^*$ and this implies $\kappa(b^*) = \kappa(b)^*$, $b \in \mathscr{B}$. ▮

5. Finite Dimension

Our aim in this section is to establish a link between our notion of conjugate and the definition of conjugate object given in [7] in any strict tensor C^*-category $\mathscr{T}$. We also establish a result concerning finite dimensional subrepresentations of tensor product representations that will be useful in the next section.

Recall that if W is an object of $\mathscr{T}$ with identity object ι, the conjugate object W_1 is defined by two intertwiners $(\iota, W_1 \times W)$ and $R_1 \in (\iota, W \times W_1)$ satisfying the conjugate equations

$$R_1^* \times 1_W \circ 1_W \times R = 1_W, \tag{5.1}$$

$$R^* \times 1_{W_1} \circ 1_{W_1} \times R_1 = 1_{W_1}, \tag{5.2}$$

where 1_W denotes the identity arrow on any object $W \in \mathscr{T}$ and $\times$ is the tensor product on the arrows.

In the particular case where $\mathscr{T} = \mathscr{T}(V)$, the tensor C^*-category of unitary corepresentations of V, we shall refer to W_1 as a strong conjugate of the object $W \in \mathscr{T}(V)$. We note explicitly that a strong conjugate W_1 of W is required to be unitary.

5.1. Lemma. *Let W and W' be corepresentations of V. If W' admits a conjugate (resp. unbounded conjugate) $\overline{W'}$, then there is a $W \times W'$-pointwise invariant one dimensional subspace if and only if there is a Hilbert–Schmidt (resp. finite rank) intertwiner from $\overline{W'}^{*-1}$ to W.*

Proof. We give a proof only in the unbounded case. A one dimensional $W \times W'$-pointwise invariant subspace is defined by a non-zero $R \in H_W \odot H_{W'}$ such that for any $\psi \in K_0$, $W_{12} W'_{13} \psi R = \psi R$. We choose orthonormal bases $\{\psi_i\}$, $\{\varphi_j\}$, $\{\varphi'_k\}$ spanning K_0, H_W and $H_{W'}$ respectively and define the finite rank operator $A\colon \bar{H}_{W'} \to H_W$ by $(\varphi_j, A\overline{\varphi'_k}) = a_{j,k}$, where $R = \Sigma\, a_{j,k} \varphi_j \varphi'_k$. It is easy to show that the above equation translates to $WI \times A\overline{W'}^* = I \times A$ on $K_0 \odot H_W \odot H_{W'}$. ▮

Let W_1 be a strong conjugate of W acting on H_1 defined by R and R_1. If $A \in L^2(\bar{H}, H_1)$ and $A_1 \in L^2(\overline{H_1}, H)$, correspond to R and R_1 respectively then (5.1) and (5.2) translate to $A_1^c A = I_{\bar{H}}$, $AA_1^c = I_{H_1}$, hence H and H_1 must be finite dimensional with the same dimension.

5.2. Proposition. *Let W be a unitary finite dimensional corepresentation of V acting on H. The following are equivalent*:

(a) *W admits a strong conjugate*;

(b) *W admits a bounded invertible conjugate $\bar{W}$ equivalent to a unitary W_1*;

(c) *W admits a bounded invertible conjugate $\bar{W}$ and $(\bar{W}^{-1*}, \bar{W})$ contains a positive invertible element.*

A strong conjugate is then defined by W_1 and is unique up to unitary equivalence.

Proof. Assume that (a) holds. Choose orthonormal bases $\{\psi_i\}$ and $\{\varphi_j\}$ of K and H respectively and define the linear operator Z on the linear span $\mathscr{V}$ of $\{\psi_i\overline{\varphi_j}\}$ by $(\psi_h\overline{\varphi_k}, Z\psi_i\overline{\varphi_j}) = (\psi_h\varphi_j, W\psi_i\varphi_k)$. If $A_R \in L^2(\bar{H}, H_1) = H_1\bar{H}$, $A_{R_1} \in L^2(\bar{H}_1, H) = HH_1$ correspond to R and R_1 respectively then by the above lemma and its proof $W_1 I \times A_R Z = I \times A_R$ on $\mathscr{V}$. Thus Z extends to an invertible operator on $K\bar{H}$ since A_R is invertible. Furthermore its adjoint is the conjugate corepresentation of W equivalent to W_1, hence (b) follows. Reversing the argument one proves similarly that (b) implies (a). If (b) holds and $B \in (W_1, \bar{W})$ is invertible then $BB^* \in (\bar{W}^{-1*}, \bar{W})$, thus we get (c). Finally if $T \in (\bar{W}^{-1*}, \bar{W})$, is positive and invertible then $W_1 = I \times T^{1/2}\bar{W} I \times T^{-1/2}$ is unitary corepresentation equivalent to $\bar{W}$, hence (c) implies (b). The rest is now clear. ∎

Note that setting $B_R = A_R{}^*$ and $B_{R_1} = A_{R_1}{}^*$, the conjugate equations (5.1) and (5.2) translate to

$$B_R \in (W_1, \bar{W}),\ B_{R_1} \in (W, \overline{W_1}),\ B_{R_1}{}^c B_R = I_{H_1}.$$

In particular, R_1 is uniquely determined by R. We define the intrinsic dimension of W relative to R by $d_R(W) = \|R\|\,\|R_1\| = \|B_R\|_{HS}\,\|B_R^{-1}\|_{HS}$, where $\|\ \|_{HS}$ denotes the Hilbert Schmidt norm. The intrinsic dimension of W, denoted $d(W)$, defined in [23] in any strict tensor C^*-category, is the infimum of all relative dimensions. It does not depend on the choice of W_1 since two strong conjugates of W are unitariliy equivalent. Clearly $d(W) = d(W_1)$. Moreover the function $W \to d(W)$ is additive and multiplicative on the semiring of unitary finite dimensional corepresentations of V gifted with strong conjugates [23]. For completeness we report from [14] the following result.

5.3. Theorem. *If W is a unitary finite dimensional corepresentation of V gifted with a strong conjugate then $d(W) \geqslant \dim(W)$. The equality holds if and only if $\bar{W}$ is unitary.*

In particular if V is discrete and regular the following are equivalent:

(a) *the natural coinverse κ of $\hat{\mathscr{A}}(V)$ is involutive*;

(b) *$d(W) = dim(W)$ for every W as above.*

If V is a regular multiplicative unitary we denote by $\mathscr{T}^0(V)$ (resp. $\mathscr{T}^0_u(V)$) the category of corepresentations W of V endowed with a conjugate (resp. unbounded conjugate) corepresentation $\bar{W}$ such that W and $\bar{W}$ are equivalent to unitary corepresentations. Note that in general both $\mathscr{T}^0(V)$ and $\mathscr{T}^0_u(V)$ are closed under equivalence, conjugation, tensor products, direct sums and finite dimensional subobjects.

If for example V is regular and discrete, that is V comes from a Woronowicz C^*-algebra, then any finite dimensional corepresentation W of V belongs to $\mathscr{T}^0(V)$, [17; Theorem 5.2], so the conjugate of W can be defined, up to equivalence, by the operators R and R_1 defined in (5.1) and (5.2). The left and right regular corepresentations V and $\vartheta W \vartheta$ are objects of $\mathscr{T}^0_u(V)$. For later use we prove the following result.

5.4. Lemma. *Let W and W' be objects of $\mathscr{T}^0(V)$ (resp. $\mathscr{T}^0_u(V)$). If one of W and W' admits no finite dimensional subcorepresentation then the same holds for the tensor product $W \times W'$.*

Proof. Let H and H' be the Hilbert spaces of W and W' respectively. To fix ideas we assume that $\bar{W}$ and $\overline{W'}$ are unbounded corepresentations and that W' admits no finite dimensional subcorepresentation. If there were a finite dimensional subobject Z contained in $W \times W'$ then by Lemma 5.1 $HH'HH'$ should contain a pointwise $\overline{W \times W'} \times W \times W'$-invariant one dimensional subspace; thus it suffices to show that HH' contains no pointwise $W \times W'$-invariant subspace. In contrary case, again by Lemma 5.1 we could find a non zero finite rank operator A intertwining $\overline{W'}^{*-1}$ with W. Now $\overline{W'}$ and W are equivalent to unitary corepresentations, therefore there is another such operator B intertwiner $\overline{W'}$ with unitary corepresentation Z_1, so B^*B is a nonzero finite rank selfintertwiner of $\overline{W'}$ and this is a contradiction. ∎

6. Actions of Hopf Algebras on Cuntz Algebras

In this section we define actions of Hopf algebras, arising from certain multiplicative unitaries, on Cuntz algebras. These actions are canonically induced by corepresentations of the underlying multiplicative unitary. If the corepresentations correspond to unitary representations of a locally compact group we get the model automorphic actions considered in [3]. We generalize results and techniques of [3] and [9].

In this section we assume that V is a regular multiplicative unitary. We associate with any unitary corepresentation W of V on a Hilbert space H the operator $S_W = W\vartheta_{H,K} \in (HK, KH)$.

If W' is another unitary corepresentation on H' then the space of intertwining operators from W to W' can be described in terms of S_W and $S_{W'}$ as

$$(W, W') = \{T \in (H, H'),\ S_{W'} T S_W^* = \sigma_K(T)\},$$

moreover

$$\begin{aligned} S_{W\times W'} &= W_{12} W'_{13} \vartheta_{HH',K} = W_{12} W'_{13} \vartheta_{H,K} \sigma_H(\vartheta_{H',K}) \\ &= W\vartheta_{H,K}\sigma_H(W'\vartheta_{H',K}) = S_W \sigma_H(S_{W'}). \end{aligned}$$

In particular for $r, s = 0, 1, 2, \ldots$,

$$(W^r, W^s) = \{T \in (H^r, H^s),\ \lambda_S(T) = \sigma_K(T)\}$$

where $S = S_W$ and $\lambda_S\colon \mathcal{O}_H \to \mathcal{O}_{SH}$ denotes the canonical isomorphism induced by S, as described in §2.

We can rephrase the fact that W is a corepresentation of V in terms of $S = S_W$ and $R = S_V$.

6.1. Proposition. *Let $W \in (KH, KH)$ be a unitary operator, then the following are equivalent*

(a) *W is a corepresentation of V;*

(b) *$R\sigma_K(S)\, S = \sigma_K(S)\, \vartheta_{H,K} \sigma_H(R)$;*

(c) *$\lambda_{\sigma_K(S)\, S} = ad(R^*) \circ \sigma_K \circ \lambda_S$ on $\mathcal{O}_H$;*

(d) *$\lambda_{S^*} \circ \lambda_{R^*} = \sigma_H \circ \lambda_{R^*}$ on $\mathcal{O}_K$.*

This proposition is a generalization of [11; Proposition 2.1], so the proof is omitted.

The C^*-subalgebra $C^*(H)$ of $\mathcal{O}_H$ generated by H is an inductive limit of extensions of finite order Cuntz algebras by the compacts, so it is nuclear since Cuntz algebras are nuclear [1]. We identify the minimal tensor product $\hat{\mathscr{A}}(V) \otimes C^*(H)$ with the Banach subspace $\hat{\mathscr{A}}(V)\, \sigma_K(C^*(H))$ of the ambient von Neumann algebra M.

6.2. Proposition. *If W is a unitary corepresentation of V on H then*

(a) *$\hat{\mathscr{A}} \lambda_S(\mathscr{K}(H^r, H^s)) + \lambda_S(\mathscr{K}(H^r, H^s))\, \hat{\mathscr{A}} \subseteq \hat{\mathscr{A}} \sigma_K(\mathscr{K}(H^r, H^s))$;*

(b) *$\lambda_S(C^*(H)) \subseteq M(\hat{\mathscr{A}} \sigma_K(C^*(H)))$;*

(c) *λ_S is a coaction of $\hat{\mathscr{A}}$ on $C^*(H)$,*

where $S = S_W$ and $\hat{\mathscr{A}} = \hat{\mathscr{A}}(V)$.

Proof. By [10; Remarque A.4] $W \in M(\hat{\mathscr{A}} \otimes \mathscr{K}(H)) \cong M(\hat{\mathscr{A}}_{\sigma_K}(\mathscr{K}(H))$, so for $\psi \in H$ and $a \in \hat{\mathscr{A}}$, $\lambda_S \psi a$ and $a \lambda_S \psi$ are elements of $\hat{\mathscr{A}} \sigma_K(H)$ so (a) and (b) follow. Now on $\hat{\mathscr{A}} \sigma_K(H) \subseteq \hat{\mathscr{A}} \otimes C^*(H)$ the monomorphism $id \otimes \lambda_S$ acts multiplying on the left by $\sigma_K(S)$, so by above it acts in the same way on $\lambda_S(H)$, that is $id_{\hat{\mathscr{A}}} \otimes \lambda_S \circ \lambda_S = \lambda_{\sigma_K(S) S}$. Similarly, $\hat{\delta} \otimes id$ acts as $ad(R^*) \circ \sigma_K$ on the image of λ_S therefore (c) follows from the previous proposition. ∎

The fixed point algebra corresponding to the coaction λ_S is

$$C^*(H)_\lambda = \{X \in C^*(H), \lambda_S(X) = \sigma_K(X)\}.$$

In particular for $r, s = 0, 1, 2, \ldots$, $\mathscr{K}(H^r, H^s) \cap C^*(H)_\lambda$ is the space $\mathscr{K}(W^{\times r}, W^{\times s})$ of compact intertwiners from $W^{\times r}$ to $W^{\times s}$.

6.3. PROPOSITION. *If $H_1 \subseteq H^r$ defines a finite dimensional subcorepresentation of $W^{\times r}$, $r \in \mathbb{N}$, then $C^*(H)_\lambda$ is σ_{H_1}-stable.*

Proof. The image of σ_K commutes with $\sigma_K(H^{r^*}) \lambda_S H^r \subseteq (K, K)$ so

$$\sigma_{\lambda_S(H_1)} \circ \sigma_K = \sigma_{\sigma_K(H_1)} \circ \sigma_K = \sigma_K \circ \sigma_{H_1}$$

being $\sigma_{H_1}(I)$ a fixed point. This implies that for $X \in C^*(H)_\lambda$,

$$\lambda_S(\sigma_{H_1}(X)) = \sigma_{\lambda_S(H_1)}(\lambda_S(X)) = \sigma_{\lambda_S(H_1)}(\sigma_K(X)) = \sigma_K(\sigma_{H_1}(X)). \quad ∎$$

If V is a discrete regular multiplicative unitary then we know from [10; Section 4] that $\hat{A}(V)$ is a Woronowicz C^*-algebra, the corresponding $\mathscr{B}$ defined in the previous section is the *-subalgebra generated by the coefficients of the finite dimensional subcorepresentations of V. If φ denotes the Haar measure of $\hat{\mathscr{A}}(V)$ then V is equivalent to a multiple of the multiplicative unitary defined in (4.2).

6.4. LEMMA. *Let V be a discrete regular multiplIcative unitary and T a normalized cofixed vector of V then the formula $\varphi(A) = T^*AT$ defines the Haar measure on $\hat{\mathscr{A}}(V)$.*

Proof. Obvious.

6.5. PROPOSITION. *Let W be a unitary corepresentation of V and assume that either W is finite dimensional or V is discrete. Then there is a conditional expectation $E: C^*(H) \to C^*(H)_\lambda$ such that $E(\mathscr{K}(H^r, H^s)) = \mathscr{K}(W^{\times r}, W^{\times s})$.*

Proof. Assume first W finite dimensional. Then the C^*-subalgebra $\hat{\mathscr{A}}(W)$ of $M(\hat{\mathscr{A}}(V))$ generated by the matrix coefficients of W, $\sigma_K(\psi^*)\, W\sigma_K(\varphi^*)$, $\psi, \varphi \in H$, is a separable unital Hopf C^*-algebra with the restricted coproduct. Furthermore a generalization of the arguments of [10; Remarques 3.11, b)] to the case where ω is a state of $\hat{\mathscr{A}}(W)$ extended to $M(\hat{\mathscr{A}}(V))$, shows that $\hat{\mathscr{A}}(W)$ is reduced to the right in the sense of [10; Paragraphe 3]. It follows that $\hat{\mathscr{A}}(W)$ admits a right invariant Haar measure m [10; Proposition 3.11.1]. Furthermore the image of λ_S is contained in $\hat{\mathscr{A}}(W)\, \sigma_K(C^*(H)) \cong \hat{\mathscr{A}}(W) \otimes C^*(H)$ since $\lambda_S(\psi) = \sum_i \sigma_K(\psi_i^*)\, W\sigma_K(\psi)\, \sigma_K(\psi_i)$, with ψ_i an orthonormal basis of H. We claim that the formula

$$\omega(E(X)) = m \circ id \otimes \omega \circ \lambda_S(X), \qquad X \in C^*(H),\ \omega \in M_*,$$

defines the desired conditional expectation. E is clearly a norm one positive map from $C^*(H)$ to the ambient von Neumann algebra M and satisfies $E(AX) = AE(X)$, $A \in C^*(H)_\lambda$, $X \in C^*(H)$ since for any $\omega \in M_*$,

$$\begin{aligned}\omega(E(AX)) &= m \circ id \otimes \omega(\sigma_K(A)\, \lambda_S(X)) \\ &= m \circ id \otimes \omega A(\lambda_S(X)) = \omega(AE(X)).\end{aligned}$$

It remains to show that if $X \in (H^r, H^s)$ then $E(X) \in (W^{\times r}, W^{\times s})$. We first show that $E(X) \in (H^r, H^s)$ and then that it is an intertwiner. Now for $\psi \in H^s$, $\varphi \in H^r$, we have: $\sigma_K(\psi^*)\, \lambda_S(X)\, \sigma_K(\varphi) \in \hat{\mathscr{A}}(W)$, so for $\omega \in M_*$,

$$\begin{aligned}\omega(\psi^* E(X)\, \varphi) = (\varphi\omega\psi^*)(E(X)) &= m \circ id \otimes \omega(\sigma_K(\psi^*)\, \lambda_S(X)\, \sigma_K(\varphi)) \\ &= m(\sigma_K(\psi^*)\, \lambda_S(X)\, \sigma_K(\varphi))\, \omega(I),\end{aligned}$$

so $\psi^* E(X)\, \varphi = m(\sigma_K(\psi^*)\, \lambda_S(X)\, \sigma_K(\varphi))\, I$ and the first claim follows. If now ψ_i and φ_j are orthonormal bases of H^s and H^r respectively, then

$$\begin{aligned}\lambda_S(E(X)) &= \sum_{i,j} m(\sigma_K(\psi_i^*)\, \lambda_S(X)\, \sigma_K(\varphi_j))\, \lambda_S(\psi_i \varphi_j^*) \\ &= m \otimes id \otimes id \circ id \otimes \lambda_S \circ \lambda_S(X) \\ &= ((m \otimes id \circ \hat{\delta}) \otimes id) \circ \lambda_S(X) \\ &= \sum_{i,j} m(\sigma_K(\psi_i^*)\, \lambda_S(X)\, \sigma_K(\varphi_j))\, \sigma_K(\psi_i \varphi_j^*) = \sigma_K(E(X))\end{aligned}$$

by the right invariance of m.

Assume now that V is discrete and let T be a normalized cofixed vector of V. We use the Haar measure induced by T on $\hat{\mathscr{A}}(V)$ to define as above a norm one positive linear map E from $C^*(H)$ to M that satisfies $E(AX) = AE(X)$, $A \in C^*(H)_\lambda$, $X \in C^*(H)$. Direct computations show that E is defined on $C^*(H)$ by the formula $E(X)=T^*\lambda_S(X)\,T$. Now $\lambda_S(\mathscr{K}(H^r, H^s))\subseteq\hat{\mathscr{A}}(V)\,\sigma_K(\mathscr{K}(H^r, H^s))$ since $\hat{\mathscr{A}}(V)$ is unital, so the image of E is contained in $C^*(H)$. The same arguments used in the finite dimensional case work and complete the proof. ∎

If W is an infinite dimensional corepresentation of a non-discrete multiplicative unitary V there is no conditional expectation onto the fixed point algebra in general. However one can show that

6.6. Proposition. *If W is any corepresentation of a regular multiplicative unitary V then the fixed point algebra $C^*(H)_\lambda$ is generated, as a Banach space, by the subspaces $\mathscr{K}(W^{\times r}, W^{\times s})$.*

Proof. By the previous proposition it suffices to assume W infinite dimensional. Let γ denote the automorphic action of $\mathbb{T}$ on $C^*(H)$ defined by $\psi\to z\psi$, $\psi\in H$, $z\in\mathbb{T}$; then the fixed point algebra is γ-stable since $\lambda_S\circ\gamma_z=id\otimes\gamma_z\circ\lambda_S$, $z\in\mathbb{T}$, so by Fourier analysis it is generated as a Banach space by its γ-eigenspaces $C^*(H)_\lambda\cap C^*(H)^k$, where

$$C^*(H)^k=\{X\in C^*(H),\ \gamma_z(X)=z^kX,\ z\in\mathbb{T}\}.$$

We now define as in [9] the projection maps

$$\pi_r^k\colon C^*(H)^k\to C^*(H)^k,\qquad r=0, 1, 2, ...,$$

acting as the identity on $H^k+\mathscr{K}(H, H^{k+1})+\cdots+\mathscr{K}(H^r, H^{r+k})$ and trivially on $\mathscr{K}(H^s, H^{s+k})$, $s>r$. For any r, π_r^0 is a unital $*$-homomorphism of the AF-C^*-algebra $C^*(H)^0$, moreover $\hat{\mathscr{A}}(V)\,\sigma_K(C^*(H)^0)$ can be identified with the minimal tensor product $\hat{\mathscr{A}}(V)\otimes C^*(H)^0$ so $id\otimes\pi_r^0$ extends on the multiplier algebra $M(\hat{\mathscr{A}}(V)\,\sigma_K(C^*(H)^0))$. Now $\lambda_S(C^*(H)^0)\subseteq M(\hat{\mathscr{A}}(V)\,\sigma_K(C^*(H)^0))$ by Proposition 6.2 and it is easy to show, using again Proposition 6.2, that $\lambda_S\circ\pi_r^0=id\otimes\pi_r^0\circ\lambda_S$ on $C^*(H)^0$. Furthermore the maps π_r^k can be recovered from the π_t^0's since for any $X\in C^*(H)^k$, $\pi_r^k(X)=\psi^*\pi_{r+k}^0(\psi X)$, with ψ a normalized vector of H^k. This shows that also $id\otimes\pi_r^k$ extends on

$$M(\hat{\mathscr{A}}(V)\,\sigma_K(C^*(H)^k))=\{T\in M,\ Ta+aT\in\hat{\mathscr{A}}(V)\,\sigma_K(C^*(H)^k),\ a\in\hat{\mathscr{A}}(V)\}$$

and that $\lambda_S\circ\pi_r^k=id\otimes\pi_r^k\circ\lambda_S$ on $C^*(H)^k$. It follows that $C^*(H)_\lambda^k$ is π_r^k-stable and the proof is complete since $\lim_r\pi_r^k(X)=X$. ∎

We now can prove the following

6.7. THEOREM. *Let W be a unitary corepresentation of V on a Hilbert space H.*

(a) *If the tensor powers $W^{\times r}$ contain no finite dimensional subcorepresentation then the coaction λ is ergodic;*

(b) *if W admits a con jugate $\bar{W}$ equivalent to a unitary corepresentation then*

$$C^*(H)_\lambda = \text{closed linear span}\{\mathcal{K}(W_1^{\times r}, W_1^{\times s}), r, s = 0, 1, 2, ...\},$$

where W_1 is the smallest subcorepresentation of W containing all the finite dimensional subcorepresentations of W. In particular λ is ergodic if and only if W has no finite dimensional subcorepresentation.

Proof. (a) is an immediate consequence of the previous proposition. Assume that the hypothesis of (b) holds and let H_1 be the space of W_1 and H_2 the orthogonal complement of H_1. Then we have the following decomposition of H^r into $W^{\times r}$-invariant subspaces $H^r = H_1^r \oplus H_1^{r-1} H_2 \oplus \cdots \oplus H_2^r$. By Lemma 5.3 any finite dimensional subcorepresentation Z of $W^{\times r}$ is contained in $W_1^{\times r}$. If $T \in \mathcal{K}(W^{\times r}, W^{\times s})$ then T^*T is a positive compact intertwiner of $W^{\times r}$ with itself so by the above argument the kernel of $T^*T - \lambda I$, $\lambda > 0$, is contained in H_1^r. This implies

$$\operatorname{Ker} T = \operatorname{Ker}(T^*T) = \left(\bigoplus_{\lambda > 0} \operatorname{Ker}(T^*T - \lambda I) \right)^\perp \supseteq (H_1^r)^\perp$$

Since $\operatorname{Ran} T^\perp = \operatorname{Ker} T^*$ we deduce that $\operatorname{Ran} T \subseteq H_1^s$. ∎

We now specialize to the case of the regular corepresentation V in order to get further information on the fixed point algebras, that we denote in this special case by $\mathcal{O}_V$. If V acts on a finite dimensional Hilbert space K we get the *canonical regular action* of the corresponding finite dimensional Hopf algebra first introduced by J. Cuntz in [11] and the corresponding fixed point algebra

$$\mathcal{O}_V = \{X \in \mathcal{O}_K, \lambda_R(X) = \sigma_K(X)\}. \tag{6.1}$$

If on the contrary K is infinite dimensional and V is non-discrete then the fixed point algebra can be very small, a result that was first noticed in the group action case [9].

6.8. COROLLARY. *If V is a non-discrete multiplicative unitary on K endowed with a bounded conjugate corepresentation $\bar{V}$ then the corresponding fixed point algebra $\mathcal{O}_V$ reduces to the complex numbers.*

Proof. Any tensor power of V is unitarily equivalent to a multiple of V itself so, by the above theorem, it suffices to show that V contains no finite dimensional subcorepresentation. If on the contrary V contained such a subcorepresentation, say Z, then $V \times \bar{V}$ should contain the trivial corepresentation; now $V \times \bar{V}$ is equivalent to a multiple of V itself, so V should contain the trivial corepresentation, and this is a contradiction. ∎

If K is allowed to be infinite dimensional and $\mathcal{O}_K$ is the generalized Cuntz algebra then (6.1) defines again a C^*-subalgebra of $\mathcal{O}_K$, denoted $\tilde{\mathcal{O}}_V$, that coincides with $\mathcal{O}_V$ if and only if K is finite dimensional. In [7] it has been shown that the C^*-subalgebra $\mathcal{D}$ of $\tilde{\mathcal{O}}_V$ generated by the intertwiners of the tensor powers of V is canonically isomorphic to the whole $\mathcal{O}_K$ if for example V comes from a locally compact group. More in general, sufficient condition are given. The existence of an isomorphism has also been proved independently by R. Longo in [12] in the case where K is finite dimensional. We report from [14] the following

6.9. Theorem. *Let V be a multiplicative unitary on a Hilbert space K, then*

(a) *if $\hat{\mathscr{A}}(V)$ is endowed with a right invariant mean then there is a conditional expectation $E: \mathcal{O}_K \to \tilde{\mathcal{O}}_V$ satisfying $E((K^r, K^s)) = (V^r, V^s)$, so $\mathcal{O}_V = \mathcal{D}$;*

if there is a unitary U on K such that the operator W defined by $\vartheta V \vartheta = I \times UWI \times U^*$ is multiplicative then

(b) *$\varepsilon = V\vartheta W$ intertwines $V^{\times 2}$ with itself;*

(c) *ε implements σ_K on (V, V);*

(d) *$\lambda_{\varepsilon^* R}$ is an isomorphism of $\mathcal{O}_K$ onto $\mathcal{D}$.*

Remark. A unitary operator U satisfying the condition required in the theorem exists if for example V is discrete, compact (and hence in the particular case where K is finite dimensional), or, more generally, if V is irreducible. If W and $\sigma_K(V)$ commute then ε induces a braided symmetry [7], [20] in the full tensor subcategory of $\mathscr{C}(V)$ generated by V. This occurs in particular when the pair (V, U) is a Kac-system [10; Proposition 6.5].

If K is infinite dimensional then the Hilbert space $H = \varepsilon^* RK \subseteq (V, V^{\times 2})$ is not contained in $\mathcal{O}_V$ since its elements are not compact operators. However we have

6.10. Theorem. *Let V be a regular discrete multiplicative unitary on K^2 and let K_0 denote the subspace of K of cofixed vectors, then with the above notations,*

(a) *the relative commutant of* $\mathcal{O}_V$ *in* $C^*(K)$ *is* $\mathbb{C}I$;

(b) $C^*(HK_0) \subset \mathcal{O}_V \subseteq C^*(H)$;

(c) $\mathcal{O}_V$ *is a purely infinite simple and nuclear* C^**-algebra.*

Proof. (a) follows from the fact that the commutant of any non zero element of K in $C^*(K)$ reduces to the compex numbers. To prove the first inclusion of (b) we note that HK_0 is a pointwise $V^{\times 2}$-invariant subspace of K^2, therefore it is contained in $\mathcal{O}_V$. To prove the second inclusion it suffices, by Proposition 6.6, that $\mathscr{K}(V^{\times r}, V^{\times s}) \subseteq C^*(H)$. We note that $\lambda_{\varepsilon^* R}$ acts identically both on K_0 and on $\mathscr{K}(V, V)$, so these subspaces are contained in $C^*(H)$. Let $E \in (V, V)$ be an orthogonal projection, then $E\varepsilon^* R = \varepsilon^* \sigma_K(E)\, R = \varepsilon^* \lambda_R(E)\, R$, thus for any $\psi \in EK$, $\varphi \in (I-E)\, K$, $\psi^* \varepsilon^* R\varphi = \psi^* \varepsilon^* \lambda_R(E\varphi) = 0$; now K is a direct sum of finite dimensional V-stable subspaces so $K^2 \subseteq H(H^* K^2) \subseteq HK$. It follows that if one of r and s, say r, is zero, then $\mathscr{K}(V^{\times r}, V^{\times s}) = H^{s-1} K_0 \subseteq H^s$. Similarly, if $r, s \neq 0$, then $\mathscr{K}(V^{\times r}, V^{\times s}) \subseteq H^{s-1} \mathscr{K}(V, V)(H^{r-1})^* \subseteq H^s (H^r)^*$, and the proof of (b) is complete. It remains to prove (c). We may assume K infinite dimensional since, otherwise, $\mathcal{O}_V$ is a Cuntz algebra of finite order, which is simple, purely infinite and nuclear [1]. As in [9] we define a state ω on $C^*(H)$ via the formula $\omega(X)\, I = \lim_i S_i^* X S_i$, where S_i is a countable orthonormal system of H. Note that $C^*(H)$ is a simple C^*-algebra containing $\mathcal{O}_V$, so if $X \neq 0$, there are elements A, B, of the dense $*$-subalgebra $\mathscr{C}$ of $C^*(H)$ generated by H such that $\omega(AXB) = I$. It follows that if T is a normalized cofixed vector of V then for sufficiently large i, $T^* S_i^* AXBS_i T$ has a distance less than 1 from the identity I. Now for sufficiently large i, $T^* S_i^* A$ and $BS_i T$ are elements of $\mathcal{O}_V$ so if $X \in \mathcal{O}_V$, then $T^* S_i^* AXBS_i T$ is invertible in $\mathcal{O}_V$, and this shows that $\mathcal{O}_V$ is simple and purely infinite. Furthermore the arguments of [21; Corollary 3.11] show that $\mathcal{O}_V$ is nuclear. ∎

We need the following general result about the corepresentation theory of a regular discrete V, easy consequence of the orthogonality conditions of [17] and [10; Appendix].

For any cardinal number m and any corepresentation W of V on H by mW we mean the tensor product of the trivial m-dimensional corepresentation with W.

6.11. Theorem. *Let V be a discrete regular multiplicative unitary and W a corepresentation of V, then*

(a) *if W is irreducible then it is finite dimensional*;

(b) *W is completely reducible*;

(c) *if $\mathscr{S}$ is a complete set of irreducible corepresentations of V then $V = \bigoplus_{W \in \mathscr{S}} m\, dW$, with d the dimension of W and m the dimension of the trivial subcorepresentation of V.*

Let $\mathscr{B}$ be a unital C^*-algebra. We denote by $\mathrm{End}(\mathscr{B})$ the tensor C^*-category of $*$-endomorphisms $\mathscr{B}$. If $\rho, \sigma \in \mathrm{End}(\mathscr{B})$ then the set of arrows from ρ to σ is

$$(\rho, \sigma) = \{A \in \mathscr{B}, A\rho(B) = \sigma(B) A, A\rho(I) = A, B \in \mathscr{B}\}.$$

We note explicitly that the endomorphisms are not assumed to be unital.

Let $\mathscr{S}_V$ denote the strict tensor C^*-category with conjugates of all the finite dimensional subcorepresentations of V. By the above theorem $\mathscr{S}_V$ contains any finite dimensional representation of the underlying Woronowicz C^*-algebra $(\hat{\mathscr{A}}(V), \hat{\delta})$. Then we can use the results of this section to define a model action of this compact quantum dual on the C^*-algebra $\mathscr{O}_V$ as follows. Let σ_W denote the restriction to $\mathscr{O}_V$ of the inner endomorphism of $C^*(H)$ induced by the Hilbert space of $W \in \mathscr{S}_V$.

6.12. THEOREM. *Let V be a discrete regular multiplicative unitary, then the map*

$$F\colon W \in \mathscr{S}_V \to \sigma_W \in \mathrm{End}(\mathscr{O}_V)$$

which acts identically on the arrows sets up a faithful functor of tensor C^*-categories with conjugates from $\mathscr{S}_V$ onto a full subcategory of End$(\mathscr{O}_V)$.

Proof. In view of the above results we have to show that the image of F is full. This follows from Theorem 6.10(a). ∎

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